CIVL222 STRENGTH OF MATERIALS

Chapter 7-Part A

Stresses in Beams – Pure Bending
Bending

• Today’s Objective:
  • Students will be able to:
    a) Determine the stress in a beam member caused by bending

In-class Activities:
• Flexural formula
• Unsymmetrical bending
• Composite Beams
• Concept Quiz
Bending Deformation of a Straight Member
Bending Deformation of a Straight Member
Bending Deformation of a Straight Member

Longitudinal “fibers”
When a bending moment is applied to a straight prismatic beam, the longitudinal lines become curved and vertical transverse lines remain straight and yet undergo a rotation.
Bending Deformation of a Straight Member

• The bottom material is stretched and the top material is compressed – leaving a neutral surface somewhere in between.

• A **neutral surface** is where longitudinal fibers of the material will not undergo a change in length.
Bending Deformation of a Straight Member

**Assumptions:**

a) Plane section remains plane
b) Length of longitudinal axis remains unchanged
c) Plane section remains perpendicular to the longitudinal axis
d) In-plane distortion of section is negligible

Before deformation

After deformation
Bending Deformation of a Straight Member

\[ \Delta x \]

\[ \Delta \theta \]

\[ M = P_a \]

Neutral surface
Bending Deformation of a Straight Member

Now lets look at a segment in isolation: Δx does not change in length but Δx′ does.

From the strain equation:

\[ \varepsilon = \lim_{\Delta x \to 0} \frac{\Delta x' - \Delta x}{\Delta x} \]

Substituting in terms of angle θ:

\[ \varepsilon = \lim_{\Delta x \to 0} \frac{(\rho - y)\Delta \theta - \rho \Delta \theta}{\rho \Delta \theta} \]

\[ \varepsilon = -\frac{y}{\rho} \]
Bending Deformation of a Straight Member

\[ \varepsilon = -\frac{y}{\rho} \]

It should be noted that longitudinal normal strain varies linearly with distance \( y \) from the neutral axis, ie:

**Hooke’s Law**

\[ \therefore \text{stress} \]

\[ \sigma_x = E \varepsilon_x \]

\[ \sigma_x = -\frac{Ey}{\rho} \]

Normal strain distribution
The Flexure Formula

The stress resultants that are related to the normal stress $\sigma_x$ acting on the cross section are

$$N = \int_A \sigma_x \, dA = -\frac{E}{\rho} \int_A y \, dA$$

$$M = -\int_A y \sigma_x \, dA = -\int_A \left( -\frac{E}{\rho} \frac{y}{2} \right) y \, dA = \frac{E}{\rho} \int_A y^2 \, dA$$

$$\int y^2 \, dA = I$$

Then

$$M = \frac{EI}{\rho}$$

**moment-curvature equation**
The Flexure Formula

But we also know that

\[ \sigma = -\frac{E}{\rho} y \]
\[ \frac{1}{\rho} = -\frac{\sigma}{yE} \]

Substitute into moment-curvature equation

\[ M = EI \left( -\frac{\sigma}{yE} \right) \]

Solve for \( \sigma \)

\[ \sigma = -\frac{My}{I} \]
Procedure for analysis

Internal moment

• Section member at point where bending or normal stress is to be determined and obtain internal moment \( M \) at the section.

• Centroidal or neutral axis for cross-section must be known since \( M \) is computed about this axis.

• If absolute maximum bending stress is to be determined, then draw moment diagram in order to determine the maximum moment in the diagram.
The Flexure Formula
The Flexure Formula

**Location of Neutral Axis**

- The $z$ and $y$ axes must pass through the centroid of the cross section.
- Therefore **NEUTRAL AXIS** Passes through the centroid of the cross section.

From statics:

$$\int_A ydA = \bar{y}A$$

$$\bar{z} = \frac{\sum_i \bar{x}_i A_i}{\sum_i A_i}, \quad \bar{y} = \frac{\sum_i \bar{y}_i A_i}{\sum_i A_i}$$
The Flexure Formula

Procedure for analysis

Section property

• Determine **moment of inertia I**, of cross-sectional area about the neutral axis.

• Methods used are discussed in Textbook Appendix A.

• Refer to the course book’s inside front cover for the values of I for several common shapes.
The Flexure Formula

**Moment of Inertia of composite area**

A composite area is made by adding or subtracting a series of “simple shaped areas” like

- **Rectangles:**
  \[ I_{\text{centroidal x-axis}} = \frac{bh^3}{12}; I_{\text{centroidal y-axis}} = \frac{hb^3}{12} \]

- **Triangles:**
  \[ I_{\text{centroidal x-axis}} = \frac{bh^3}{36}; I_{\text{centroidal y-axis}} = \frac{hb^3}{36} \]

- **Circles:**
  \[ I_{\text{centroidal x-axis}} = I_{\text{centroidal y-axis}} = \frac{\pi r^4}{4} \text{ due to symmetry} \]

**Parallel-axis theorem**

\[ I_y = \bar{I}_y + Ad_x^2 \]
\[ I_x = \bar{I}_x + Ad_y^2 \]
The Flexure Formula

Procedure for analysis

Normal stress

• Specify distance $y$, measured perpendicular to neutral axis to point where normal stress is to be determined.

• Apply equation $\sigma = -\frac{My}{I}$, or if maximum bending stress is needed, use $\sigma_{\text{max}} = \frac{Mc}{I}$.

• Ensure units are consistent when substituting values into the equations
The Flexure Formula

\[ \sigma_x = -\frac{M y}{I_z} \]

\( \sigma_x \) stress at any point \( y \) on the cross section

(a) Bending stresses caused by positive \( M \)

(b) Bending stresses caused by negative \( M \)
Example #1

A beam has a rectangular cross section and is subjected to the stress distribution shown in Fig. 6–27a. Determine the internal moment $M$ at the section caused by the stress distribution $(a)$ using the flexure formula, $(b)$ by finding the resultant of the stress distribution using basic principles.
**Example #1 (Continued)**

**SOLUTION**

**Part (a).** The flexure formula is \( \sigma_{\text{max}} = \frac{Mc}{I} \). From Fig. 6–27a, \( c = 60 \text{ mm} \) and \( \sigma_{\text{max}} = 20 \text{ MPa} \). The neutral axis is defined as line \( NA \), because the stress is zero along this line. Since the cross section has a rectangular shape, the moment of inertia for the area about \( NA \) is determined from the formula for a rectangle given on the inside front cover; i.e.,

\[
I = \frac{1}{12}bh^3 = \frac{1}{12}(60 \text{ mm})(120 \text{ mm})^3 = 864(10^4) \text{ mm}^4
\]

Therefore,

\[
\sigma_{\text{max}} = \frac{Mc}{I}, \quad 20 \text{ N/mm}^2 = \frac{M(60 \text{ mm})}{864(10^4) \text{ mm}^4}
\]

\[
M = 288(10^4) \text{ N} \cdot \text{mm} = 2.88 \text{ kN} \cdot \text{m}
\]
Part (b). First we will show that the resultant force of the stress distribution is zero. As shown in Fig. 6–27b, the stress acting on the arbitrary element strip \( dA = (60 \text{ mm}) \, dy \), located \( y \) from the neutral axis, is

\[
\sigma = \left( \frac{-y}{60 \text{ mm}} \right) (20 \text{ N/mm}^2)
\]
Example #1 (Continued)

The force created by this stress is $dF = \sigma \, dA$, and thus, for the entire cross section,

$$F_R = \int_A \sigma \, dA = \int_{-60 \text{ mm}}^{+60 \text{ mm}} \left( \frac{-y}{60 \text{ mm}} \right)(20 \text{ N/mm}^2) \right] (60 \text{ mm}) \, dy$$

$$= (-10 \text{ N/mm}^2) \, y^2 \left|_{-60 \text{ mm}}^{+60 \text{ mm}} \right. = 0$$

The resultant moment of the stress distribution about the neutral axis ($z$ axis) must equal $M$. Since the magnitude of the moment of $dF$ about this axis is $dM = y \, dF$, and $dM$ is always positive, Fig. 6–27b, then for the entire area,

$$M = \int_A y \, dF = \int_{-60 \text{ mm}}^{+60 \text{ mm}} y \left( \frac{y}{60 \text{ mm}} \right)(20 \text{ N/mm}^2) \right] (60 \text{ mm}) \, dy$$

$$= \left( \frac{20}{3} \text{ N/mm}^2 \right) y^3 \left|_{-60 \text{ mm}}^{+60 \text{ mm}} \right. = 288(10^4) \text{ N} \cdot \text{mm} = 2.88 \text{ kN} \cdot \text{m} \quad \text{Ans.}$$
The above result can also be determined without the need for integration. The resultant force for each of the two triangular stress distributions in Fig. 6–27c is graphically equivalent to the volume contained within each stress distribution. Thus, each volume is

\[ F = \frac{1}{2} (60 \text{ mm})(20 \text{ N/mm}^2)(60 \text{ mm}) = 36(10^3) \text{ N} = 36 \text{ kN} \]

\[ M = F_C \times 80 = 36000 \times 80 = 2880000 \text{ N.mm} \]
Example #2

A beam having a tee-shaped cross section is subjected to equal **18 kN.m** bending moments. As shown in left figure. The cross-sectional dimensions of the beam are shown in right figure. Determine:

(a) The centroid location, the moment of inertia about the **z axis**, and the controlling section modulus about the **z axis**.

(b) The bending stress at point **H**, state whether the normal stress at **H** is tension or compression.

(c) The maximum bending stress produced in the cross section. State whether the stress is tension or compression.
Unsymmetric Bending

Moment applied along principal axis

\[ F_R = \Sigma F_x \quad 0 = \int \sigma \, dA \]

\[ (M_R)_y = \Sigma M_y \quad 0 = \int z \sigma \, dA \]

\[ (M_R)_z = \Sigma M_z \quad 0 = \int -y \sigma \, dA \]

If \( y \) and \( z \) are the principal axes. \( \int yz \, dA = 0 \)

(The integral is called the product of inertia)
Unsymmetric Bending

Moment arbitrarily applied

\[ \sigma = - \frac{M_z y}{I_z} + \frac{M_y z}{I_y} \]
Alternatively, identify the orientation of the principal axes (of which one is the neutral axis). Orientation of neutral axis:

\[
\tan \alpha = \frac{I_z}{I_y} \tan \theta
\]
Note: The angle $\alpha$, which is measured positive from the $+z$ axis toward the $+y$ axis, will lie between the line of action of $M$ and the $y$ axis; i.e., $\theta \leq \alpha \leq 90^\circ$. 
Example #3

The rectangular cross section shown in Fig. 6–35a is subjected to a bending moment of $M = 12 \text{ kN} \cdot \text{m}$. Determine the normal stress developed at each corner of the section, and specify the orientation of the neutral axis.
Example #3 (Continued)

Internal Moment Components. By inspection it is seen that the $y$ and $z$ axes represent the principal axes of inertia since they are axes of symmetry for the cross section. As required we have established the $z$ axis as the principal axis for maximum moment of inertia. The moment is resolved into its $y$ and $z$ components, where

\[ M_y = -\frac{4}{5}(12 \text{ kN} \cdot \text{m}) = -9.60 \text{ kN} \cdot \text{m} \]

\[ M_z = \frac{3}{5}(12 \text{ kN} \cdot \text{m}) = 7.20 \text{ kN} \cdot \text{m} \]

Section Properties. The moments of inertia about the $y$ and $z$ axes are

\[ I_y = \frac{1}{12}(0.4 \text{ m})(0.2 \text{ m})^3 = 0.2667(10^{-3}) \text{ m}^4 \]

\[ I_z = \frac{1}{12}(0.2 \text{ m})(0.4 \text{ m})^3 = 1.067(10^{-3}) \text{ m}^4 \]
Example #3 (Continued)

**Bending Stress.** Thus,

\[ \sigma = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y} \]

\[ \sigma_B = -\frac{7.20 \times 10^3 \text{ N} \cdot \text{m}(0.2 \text{ m})}{1.067 \times 10^{-3} \text{ m}^4} + \frac{-9.60 \times 10^3 \text{ N} \cdot \text{m}(-0.1 \text{ m})}{0.2667 \times 10^{-3} \text{ m}^4} = 2.25 \text{ MPa} \quad \text{Ans.} \]

\[ \sigma_C = -\frac{7.20 \times 10^3 \text{ N} \cdot \text{m}(0.2 \text{ m})}{1.067 \times 10^{-3} \text{ m}^4} + \frac{-9.60 \times 10^3 \text{ N} \cdot \text{m}(0.1 \text{ m})}{0.2667 \times 10^{-3} \text{ m}^4} = -4.95 \text{ MPa} \quad \text{Ans.} \]

\[ \sigma_D = -\frac{7.20 \times 10^3 \text{ N} \cdot \text{m}(-0.2 \text{ m})}{1.067 \times 10^{-3} \text{ m}^4} + \frac{-9.60 \times 10^3 \text{ N} \cdot \text{m}(0.1 \text{ m})}{0.2667 \times 10^{-3} \text{ m}^4} = -2.25 \text{ MPa} \quad \text{Ans.} \]

\[ \sigma_E = -\frac{7.20 \times 10^3 \text{ N} \cdot \text{m}(-0.2 \text{ m})}{1.067 \times 10^{-3} \text{ m}^4} + \frac{-9.60 \times 10^3 \text{ N} \cdot \text{m}(-0.1 \text{ m})}{0.2667 \times 10^{-3} \text{ m}^4} = 4.95 \text{ MPa} \quad \text{Ans.} \]

The resultant normal-stress distribution has been sketched using these values, Fig. 6–35b. Since superposition applies, the distribution is linear as shown.
**Orientation of Neutral Axis.** The location \( z \) of the neutral axis (NA), Fig. 6–35b, can be established by proportion. Along the edge \( BC \), we require

\[
\frac{2.25 \text{ MPa}}{z} = \frac{4.95 \text{ MPa}}{(0.2 \text{ m} - z)}
\]

\[
0.450 - 2.25z = 4.95z
\]

\[
z = 0.0625 \text{ m}
\]

In the same manner this is also the distance from \( D \) to the neutral axis in Fig. 6–35b.
Example #3 (Continued)

In the same manner this is also the distance from $D$ to the neutral axis in Fig. 6–35b.
We can also establish the orientation of the NA using Eq. 6–19, which is used to specify the angle $\alpha$ that the axis makes with the $z$ or maximum principal axis. According to our sign convention, $\theta$ must be measured from the $+z$ axis toward the $+y$ axis. By comparison, in Fig. 6–35c, $\theta = -\tan^{-1} \frac{4}{3} = -53.1^\circ$ (or $\theta = +306.9^\circ$).

\[ M = 12 \text{ kN}\cdot\text{m} \]
Example #3 (Continued)

Thus,

\[
\tan \alpha = \frac{I_z}{I_y} \tan \theta
\]

\[
\tan \alpha = \frac{1.067 \times 10^{-3} \text{ m}^4}{0.2667 \times 10^{-3} \text{ m}^4} \tan(-53.1^\circ)
\]

\[
\alpha = -79.4^\circ \quad \text{Ans.}
\]

This result is shown in Fig. 6–35c. Using the value of \( \gamma \) calculated above, verify, using the geometry of the cross section, that one obtains the same answer.
Composite Beams

APPLICATIONS

Reinforced concrete beam

Sandwich panel
Composite Beams

Transformed homogeneous beam obtained through a transformation factor:

\[ n = \frac{E_1}{E_2} \]

and

\[ dF = \sigma \, dA = \sigma' \, dA' \]

\[ \sigma \, dz \, dy = \sigma' \, n \, dz \, dy \]

\[ \sigma = n \, \sigma' \]
Example #4

A composite beam is made of wood and reinforced with a steel strap located on its bottom side. It has the cross-sectional area shown in Fig. 6–40a. If the beam is subjected to a bending moment of $M = 2 \text{ kN} \cdot \text{m}$, determine the normal stress at points $B$ and $C$. Take $E_w = 12 \text{ GPa}$ and $E_{st} = 200 \text{ GPa}$. 

![Diagram of the composite beam with dimensions and bending moment](image)
**Section Properties.** Although the choice is arbitrary, here we will transform the section into one made entirely of steel. Since steel has a greater stiffness than wood \((E_{st} > E_w)\), the width of the wood must be *reduced* to an equivalent width for steel. Hence \(n\) must be less than one. For this to be the case, \(n = E_w/E_{st}\), so that

\[
b_{st} = nb_w = \frac{12 \text{ GPa}}{200 \text{ GPa}} \times (150 \text{ mm}) = 9 \text{ mm}
\]

The transformed section is shown in Fig. 6–40b.
Example #4 (Continued)

The location of the centroid (neutral axis), computed from a reference axis located at the bottom of the section, is

\[
\bar{y} = \frac{\sum yA}{\sum A} = \frac{[0.01 \text{ m}](0.02 \text{ m})(0.150 \text{ m}) + [0.095 \text{ m}](0.009 \text{ m})(0.150 \text{ m})}{0.02 \text{ m}(0.150 \text{ m}) + 0.009 \text{ m}(0.150 \text{ m})} = 0.03638 \text{ m}
\]

The moment of inertia about the neutral axis is therefore

\[
I_{NA} = \left[ \frac{1}{12} (0.150 \text{ m})(0.02 \text{ m})^3 + (0.150 \text{ m})(0.02 \text{ m})(0.03638 \text{ m} - 0.01 \text{ m})^2 \right]
+ \left[ \frac{1}{12} (0.009 \text{ m})(0.150 \text{ m})^3 + (0.009 \text{ m})(0.150 \text{ m})(0.095 \text{ m} - 0.03638 \text{ m})^2 \right]
= 9.358(10^{-6}) \text{ m}^4
\]
**Example #4 (Continued)**

*Normal Stress.* Applying the flexure formula, the normal stress at $B'$ and $C$ is

$$
\sigma_{B'} = \frac{2 \text{ kN} \cdot \text{m}(0.170 \text{ m} - 0.03638 \text{ m})}{9.358 \times 10^{-6} \text{ m}^4} = 28.6 \text{ MPa}
$$

$$
\sigma_C = \frac{2 \text{ kN} \cdot \text{m}(0.03638 \text{ m})}{9.358 \times 10^{-6} \text{ m}^4} = 7.78 \text{ MPa}
$$

*Ans.*
The normal-stress distribution on the transformed (all steel) section is shown in Fig. 6–40c.

The normal stress in the wood, located at \( B \) in Fig. 6–40a, is determined from Eq. 6–21; that is,

\[
\sigma_B = n\sigma_{B'} = \frac{12 \text{ GPa}}{200 \text{ GPa}} (28.56 \text{ MPa}) = 1.71 \text{ MPa}
\]

\textbf{Ans.}
Using these concepts, show that the normal stress in the steel and the wood at the point where they are in contact is $\sigma_{st} = 3.50$ MPa and $\sigma_w = 0.210$ MPa, respectively. The normal-stress distribution in the actual beam is shown in Fig. 6–40d.
Example #5

The reinforced concrete beam has the cross-sectional area shown in Fig. 6–43a. If it is subjected to a bending moment of $M = 60 \text{ kN} \cdot \text{m}$, determine the normal stress in each of the steel reinforcing rods and the maximum normal stress in the concrete. Take $E_{st} = 200 \text{ GPa}$ and $E_{conc} = 25 \text{ GPa}$.
Example #5 (Continued)

Since the beam is made from concrete, in the following analysis we will neglect its strength in supporting a tensile stress.

**Section Properties.** The total area of steel, \( A_{st} = 2[\pi (12.5 \text{ mm})^2] = 982 \text{ mm}^2 \) will be transformed into an equivalent area of concrete, Fig. 6-43b. Here

\[
A' = nA_{st} = \frac{200(10^3) \text{ MPa}}{25(10^3) \text{ MPa}} \times (982 \text{ mm}^2) = 7856 \text{ mm}^2
\]
We require the centroid to lie on the neutral axis. Thus $\Sigma \bar{y}A = 0$, or

$$300 \text{ mm}(h') \frac{h'}{2} - 7856 \text{ mm}^2 (400 \text{ mm} - h') = 0$$

$$h'^2 + 52.37h' - 20949.33 = 0$$

Solving for the positive root,

$$h' = 120.90 \text{ mm}$$

Using this value for $h'$, the moment of inertia of the transformed section, computed about the neutral axis, is

$$I = \left[ \frac{1}{12} (300 \text{ mm})(120.90 \text{ mm})^3 + 300 \text{ mm}(120.90 \text{ mm}) \left( \frac{120.9 \text{ mm}}{2} \right)^2 \right] + 7856 \text{ mm}^2 (400 \text{ mm} - 120.90 \text{ mm})^2$$

$$= 788.67 \times 10^6 \text{ mm}^4$$
Example #5 (Continued)

**Normal Stress.** Applying the flexure formula to the transformed section, the maximum normal stress in the concrete is

\[
(\sigma_{\text{conc}})_{\text{max}} = \frac{60 \text{ kN} \cdot \text{m} \cdot (1000 \text{ mm/m})(120.90 \text{ mm})(1000 \text{ N/kN})}{788.67 \times 10^6 \text{ mm}^4} = 9.20 \text{ MPa} \quad \text{Ans.}
\]

The normal stress resisted by the “concrete” strip, which replaced the steel, is

\[
\sigma'_{\text{conc}} = \frac{60 \text{ kN} \cdot \text{m} \cdot (1000 \text{ mm/m})(1000 \text{ N/kN})(400 \text{ mm} - 120.9 \text{ mm})}{788.67 \times 10^6 \text{ mm}^4} = 21.23 \text{ MPa}
\]

The normal stress in each of the two reinforcing rods is therefore

\[
\sigma_{st} = n\sigma'_{\text{conc}} = \left(\frac{200(10^3) \text{ MPa}}{25(10^3) \text{ MPa}}\right)21.23 \text{ MPa} = 169.84 \text{ MPa} \quad \text{Ans.}
\]
Example #5 (Continued)

The normal-stress distribution is shown graphically in Fig. 6–43c.
1) Provided that the bending formation of a straight member is small and within elastic range. Which of the following statements is incorrect?

A) Plane section remains plane  
B) Cross section remains perpendicular 
C) The length of the longitudinal axis remains unchanged  
D) In-plane distortion of cross section is to the longitudinal axis not negligible

2) Which of the following statements is incorrect for bending of a straight member?

A) Bending stress is proportional to the moment  
B) Bending stress is inversely proportional to the second moment of area of the section  
C) Bending stress is inversely proportional to the moment of inertia of the section  
D) Bending stress is not a function of the location
1) Which of the following statements is true?

The flexure formula for a straight member can be applied only

A) when bending occurs about axes that represent the principal axes of inertia for the section.
B) The principal axes have their origin at the centroid.
C) The principal axes are orientated along an axis of symmetry, if there is one, and perpendicular to it.
D) All of the above.