<table>
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<tr>
<th>CIVL 222 STRENGTH OF MATERIALS</th>
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Terminology

1. Displacement
2. Deformation
3. Strain
4. Average Axial Strain
5. Shearing Strain
6. Poisson’s Ratio
7. Mechanical Properties of Materials
8. Biaxial Stresses and Strains
9. Triaxial Stress and Volumetric Strain
1. Displacement

Movement of a point w.r.t. a reference system. Maybe caused by translation and or rotation of object (rigid body).

Change in shape or size related to displacements are called deformations. Change in linear dimension causes deformation $\delta$. 
2. Deformation

Includes changes in both **lengths** and **angles**.

a) The un-deformed bar

b) The deformed bar
3. Strain

**Stress** is used to measure the intensity of internal force.

**Strain** is the quantity that is used to measure the intensity of deformation.

**Normal strain**, $\varepsilon$, used to measure change in size.

**Shear strain**, $\gamma$, used to measure change in shape.
4. Average Axial Strain

\[ \varepsilon = \varepsilon_{\text{avg}} = \frac{\Delta L}{L} = \frac{\delta_n}{L} \]
Definition of Extensional Strain

\[ \varepsilon = \varepsilon_{\text{avg}} = \frac{\Delta L}{L} \]

Units of Strain

Units of strain are length per length.

Technically strain is non-dimesional.

Normal strains - expressed as inch per inch (in/in)

Shear strains - expressed as radians or micro-radians.
Definition of Average Extensional Strain

\[ \varepsilon_{\text{avg}} = \frac{\Delta L}{L} = \frac{L' - L}{L} \]

If the bar stretches \((L' > L)\),
the strain is positive and called a \textbf{tensile strain}.

If the bar contracts \((L' < L)\),
the strain is negative and called a \textbf{compressive strain}.
Normal Strain

\[ \sigma = \frac{P}{A} = \text{stress} \]
\[ \varepsilon = \frac{\delta}{L} = \text{normal strain} \]

\[ \sigma = \frac{2P}{2A} = \frac{P}{A} \]
\[ \varepsilon = \frac{\delta}{L} \]

\[ \sigma = \frac{P}{A} \]
\[ \varepsilon = \frac{2\delta}{2L} = \frac{\delta}{L} \]
Thermal Strain

When a body is subjected to a temperature increase, $\Delta T$, then it will expand in all directions.

$$\varepsilon_T = \alpha \Delta T$$

$\varepsilon_T$  **Thermal Strain**

$\alpha$  **Coefficient of Thermal Expansion**

$\Delta T$  **Temperature Change**
Free thermal expansion of a uniform block caused by a uniform temperature increases the strain.

\[
\varepsilon_{xT} = \varepsilon_{yT} = \varepsilon_{zT} = \alpha \Delta T
\]

\[
\Delta L_x = (\alpha \Delta T) L_x
\]

\[
\Delta L_y = (\alpha \Delta T) L_y
\]

\[
\Delta L_z = (\alpha \Delta T) L_z
\]

\[\alpha = \text{coefficient of thermal expansion}\]
\[\Delta T = \text{change in temperature}\]
\[\Delta L = \text{change in length}\]
5. Shearing Strain

• The shearing strain may be expressed thus:  \[ \varepsilon_s = \frac{\Delta}{h} = \tan \gamma \]

• Since only small strains occur in structural components, \( \gamma \) will be small, so that:
  \[ \tan \gamma = \gamma \quad \text{radians} \]

• The shearing strain is therefore:  \[ \varepsilon_s = \gamma \quad \text{radians} \]
6. Poisson’s Ratio

- When body subjected to axial tensile force, it elongates and contracts laterally.

- Similarly, it will contract and its sides expand laterally when subjected to an axial compressive force.
6. Poisson’s Ratio

- Strains of the bar are:

\[ \varepsilon_{\text{long}} = \frac{\delta}{L} \quad \varepsilon_{\text{lat}} = \frac{\delta'}{r} \]

- Early 1800’s, S.D. Poisson realized that within elastic range, ratio of the two strains is a constant value, since both are proportional.

\[ \text{Poisson’s ratio, } \nu = -\frac{\varepsilon_{\text{lat}}}{\varepsilon_{\text{long}}} \]
6. Poisson’s Ratio

• \( \nu \) is unique for homogenous and isotropic material

• **Why negative sign?** Longitudinal elongation cause lateral contraction (-ve strain) and vice versa

• Lateral strain is the same in all lateral (radial) directions

• Poisson’s ratio is dimensionless, \( 0 \leq \nu \leq 0.5 \)
Poisson’s ratio, \( \nu = -\frac{\varepsilon_{\text{lat}}}{\varepsilon_{\text{long}}} \)
7.1 Tension and Compression Test

• Strength of a material can only be determined by experiment

• One test used by engineers is the tension or compression test

• This test is used primarily to determine the relationship between the average normal stress and average normal strain in common engineering materials, such as metals, ceramics, polymers and composites
Performing the tension or compression test

- Specimen of material is made into “standard” shape and size
- Before testing, 2 small punch marks identified along specimen’s length
- Measurements are taken of both specimen’s initial cross-sectional area $A_0$ and gauge-length distance $L_0$; between the two marks

$d_0 = 13 \text{ mm}$

$L_0 = 50 \text{ mm}$
Stress-Strain Test

Seat the specimen into a testing machine shown below.

Fig. 2.7 This machine is used to test tensile test specimens, such as those shown in this chapter.

Fig. 2.8 Test specimen with tensile load.
7.1 Tension and Compression Test

- The machine will stretch specimen at slow constant rate until breaking point
- At frequent intervals during test, data is recorded of the applied load $P$. 

![Equipment Diagram]

**Specimen**

- movable upper crosshead
- tension specimen
- load dial
- motor and load controls

$\text{Specimen}$

$\text{Equipment}$

$d_0 = 13 \text{ mm}$

$L_0 = 50 \text{ mm}$
7.1 Tension and Compression Test

- Elongation $\delta = L - L_0$ is measured using either a caliper or an extensometer.
- $\delta$ is used to calculate the normal strain in the specimen.
- Sometimes, strain can also be read directly using an electrical-resistance strain gauge.
7.2 Stress-Strain Diagram

• A stress-strain diagram is obtained by plotting the various values of the stress and corresponding strain in the specimen

Conventional stress-strain diagram

• Using recorded data, we can determine nominal or engineering stress by

\[ \sigma = \frac{P}{A_0} \]

Assumption: Stress is constant over the cross-section and throughout region between gauge points
Conventional Stress-Strain Diagram

• Likewise, nominal or engineering strain is found directly from strain gauge reading, or by

\[ \varepsilon = \frac{\delta}{L_0} \]

**Assumption:** Strain is constant throughout region between gauge points

By plotting \( \sigma \) (ordinate) against \( \varepsilon \) (abscissa), one can get a conventional stress-strain diagram
7.2 Stress-Strain Diagram

Note the critical status for strength specification:

- proportional limit
- elastic limit
- yield stress
- ultimate stress
- fracture stress

Conventional and true stress-strain diagrams for ductile material (steel) (not to scale)
Elastic behavior.

- A straight line
- Stress is proportional to strain, i.e., linearly elastic
- Upper stress limit, or proportional limit; $\sigma_{pl}$

- If load is removed upon reaching elastic limit, specimen will return to its original shape
Yielding.

- **Material deforms permanently; yielding; plastic deformation**
- **Yield stress,** \( \sigma_Y \)
- **Once yield point reached, specimen continues to elongate (strain) without any increase in load**
- **Material is referred to as being perfectly plastic**
Strain hardening.

- Ultimate stress, $\sigma_u$
- While specimen is elongating, its cross-sectional area will decrease

- Decrease in area is fairly uniform over entire gauge length

7.2 Stress-Strain Diagram

Conventional and true stress-strain diagrams for ductile material (steel) (not to scale)
Necking.

- At ultimate stress, cross-sectional area begins to decrease in a localized region.

- As a result, a constriction or “neck” tends to form in this region as specimen elongates further.
7.2 Stress-Strain Diagram

Necking.

- Specimen finally breaks at fracture stress, $\sigma_f$
7.2 Stress-Strain Diagram

**True stress-strain diagram**

- Instead of using *original* cross-sectional area and length, we can use the actual cross-sectional area and length at the *instant* the load is measured.

- In strain-hardening range, conventional $\sigma$-$\varepsilon$ diagram shows specimen supporting *decreasing load*.

- While true $\Sigma$-$\varepsilon$ diagram shows material to be sustaining *increasing stress*.
Stress-Strain Diagram: Ductile Materials

(a) Low-carbon steel

(b) Aluminum alloy
Stress-Strain Diagram: Brittle Materials

Stress-Strain diagram for a typical brittle material

\( \sigma_U = \sigma_B \)

Rupture

(a) Stress-strain diagrams.

Brittle material

Ductile material
Stress-Strain Diagram: Brittle Materials

Compression of a Concrete Cylinder
Hooke’s Law: Modulus of Elasticity

- Below the yield stress
  \[ \sigma = E \varepsilon \]
  \[ E = \text{Young's Modulus or Modulus of Elasticity} \]

- Strength is affected by alloying, heat treating, and manufacturing process but stiffness (Modulus of Elasticity) is not.

- Most grades of steel have same modulus of elasticity, \( E_{st} = 200 \text{ GPa} \)

Stress-Strain diagrams for iron and different grades of steel

- Quenched, tempered alloy steel (A709)
- High-strength, low-alloy steel (A992)
- Carbon steel (A36)
- Pure iron
Elastic vs. Plastic Behavior

- If the strain disappears when the stress is removed, the material is said to behave **elastically**.
- The largest stress for which this occurs is called the **elastic limit**.
- When the strain does not return to zero after the stress is removed, the material is said to behave **plastically**.
In order to simplify the nonlinearity of stress-strain relation, approximate simple models may be used. Among these simple models mostly used ones are given below:

- **Linear elastic materials**
- **Rigid-ideal plastic material**
- **Rigid-hardening plastic material**
- **Linear elastic-plastic material**
- **Hardening elastic-plastic material**
8. Biaxial Stresses and Strains

\[ \Delta A = B \Delta L + L \Delta B + \Delta L \Delta B \]
8. Biaxial Stresses and Strains

<table>
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<th>$\sigma_y$</th>
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<tr>
<td>Strain in direction of $\sigma_x$</td>
<td>$\frac{\sigma_x}{E}$</td>
<td>$-\frac{\nu \sigma_y}{E}$</td>
</tr>
<tr>
<td>Strain in direction of $\sigma_y$</td>
<td>$-\frac{\nu \sigma_x}{E}$</td>
<td>$\frac{\sigma_y}{E}$</td>
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- The total strain in the direction of $\sigma_x$ \( \varepsilon_x = \left( \sigma_x - \nu \sigma_y \right) / E \)
- The total strain in the direction of $\sigma_y$ \( \varepsilon_y = \left( \sigma_y - \nu \sigma_x \right) / E \)
- The change in area: \( \Delta A = B \Delta L + L \Delta B + \Delta L \Delta B \)
- The areal strain is therefore: \( \frac{\Delta A}{A} = \frac{B \Delta L + L \Delta B}{BL} = \frac{\Delta L}{L} + \frac{\Delta B}{B} \)

- For small strains, the areal strain is equal to the sum of the biaxial linear strain \( \varepsilon_A = \varepsilon_x + \varepsilon_y \)
9. Triaxial Stress and Volumetric Strain
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<td>$\frac{\sigma_y}{E}$</td>
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</tr>
<tr>
<td>Strain in direction of $\sigma_z$</td>
<td>$-\frac{\nu \sigma_x}{E}$</td>
<td>$-\frac{\nu \sigma_y}{E}$</td>
<td>$\frac{\sigma_z}{E}$</td>
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- The total strain in the direction of $\sigma_x$ is:
  \[ \varepsilon_x = \frac{\sigma_x - \nu (\sigma_y + \sigma_z)}{E} \]

- The total strain in the direction of $\sigma_y$ is:
  \[ \varepsilon_y = \frac{\sigma_y - \nu (\sigma_x + \sigma_z)}{E} \]

- The total strain in the direction of $\sigma_z$ is:
  \[ \varepsilon_z = \frac{\sigma_z - \nu (\sigma_x + \sigma_y)}{E} \]

- The Volumetric strain is therefore:
  \[ \varepsilon_v = \frac{\Delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z \]
Dilatation: Bulk Modulus

- Relative to the unstressed state, the change in volume is
  \[ \varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z \]
  \[ \varepsilon_v = \frac{1-2\nu}{E} (\sigma_x + \sigma_y + \sigma_z) \]
  = dilatation (change in volume per unit volume)

- For element subjected to uniform hydrostatic pressure,
  \[ \varepsilon_v = -p \frac{3(1-2\nu)}{E} = -\frac{p}{k} \]
  \[ k = \frac{E}{3(1-2\nu)} = \text{bulk modulus} \]

- Subjected to uniform pressure, dilatation must be negative, therefore
  \[ 0 < \nu < \frac{1}{2} \]
Deformations Under Axial Loading

- From Hooke’s Law:
  \[ \sigma = E\varepsilon \quad \varepsilon = \frac{\sigma}{E} = \frac{P}{AE} \]

- From the definition of strain:
  \[ \varepsilon = \frac{\delta}{L} \]

- Equating and solving for the deformation,
  \[ \delta = \frac{PL}{AE} \]

- With variations in loading, cross-section or material properties,
  \[ \delta = \sum_i \frac{P_i L_i}{A_i E_i} \]
An axially loaded slender bar will elongate in the axial direction and contract in the transverse directions.

An initially cubic element oriented as in top figure will deform into a rectangular parallelepiped. The axial load produces a normal strain.

If the cubic element is oriented as in the bottom figure, it will deform into a rhombus. Axial load also results in a shear strain.

Components of normal and shear strain are related,

\[ \frac{E}{2G} = (1 + \nu) \]
Example #1:

A steel bar of rectangular cross-section **100 mm x 40 mm** is subjected to an axial tension of **240 kN**. Determine the changes that result in the cross-section dimensions. **$E = 200 kN/mm^2$**, Poisson’s ratio **$n = 0.3$**
Example #2:

The rigid bar $AC$ in figure is supported by two axial bars (1) and (2). Both axial bars are made of bronze [$E = 100$ GPA; $\alpha = 18 \times 10^{-6}$ mm/mm/°C]. The cross-sectional area of bar (1) is $A_1 = 240$ mm$^2$ and the cross-sectional area of bar (2) is $A_2 = 360$ mm$^2$. After load $P$ has been applied and the temperature of the entire assembly has increased by 20 °C. The total strain in bar (2) is measured as 800 m/m (elongation).

Determine:

(a) The magnitude of load $P$,

(b) the vertical displacement of pin $A$. 
Example #3:

The figure below shows a flat steel panel which is subjected to biaxial tensile stresses. Determine the value of $s_y$ at which the strain in this direction ($e_y$) will be zero. What increase will have taken place in the 1.80 m dimension? Young’s modulus = 210 kN/mm$^2$, Poisson’s ratio $n = 0.3$
Example #4:

The 4-mm-diameter cable BC is made of a steel with $E = 200 \text{ GPA}$. Knowing that the maximum stress in the cable must not exceed 190 MPa and that the elongation of the cable must not exceed 6 mm, find the maximum load $P$ that can be applied as shown.
Example #5:

The steel frame \((E = 200 \text{ GPA})\) shown has a diagonal brace \(BD\) with an area of 1920 mm\(^2\). Determine the largest allowable load \(P\) if the change in length of member \(BD\) is not to exceed 1.6 mm.
Example #6:

A rigid bar $ABCD$ is supported by two bars as shown in Fig. P2.4. There is no strain in the vertical bars before load $P$ is applied. After load $P$ is applied, the normal strain in rod (1) is $-570 \, \mu \text{m/m}$. Determine:

(a) the normal strain in rod (2).

(b) the normal strain in rod (2) if there is a 1-mm gap in the connection at pin $C$ before the load is applied.

(c) the normal strain in rod (2) if there is a 1-mm gap in the connection at pin $B$ before the load is applied.
Example #7:

The two member frame is subjected to the distributed loading shown. Determine the average normal stress and average shear stress acting at the sections a-a and b-b. Member CB has square section of 30mm on each side. Take $w = 6\text{kN/m}$.
Example #7: Quiz No:1

A 12mm diameter steel rod AB is fitted to a round hole near end C of the wooden member CD. For the loads shown, determine

a) the maximum average normal stress in the wood
b) the distance b for which the average shearing stress is 620 kPa on the surface indicated by the dashed lines

c) the average bearing stress on the wood.
The tensile test was conducted on a mild steel specimen. The following data were obtained from the test:

Diameter of specimen \((d)\) : 25 mm
Length of specimen \((L_o)\) : 300 mm
Load at proportionality limit \((P_y)\) : 127.65 kN
Extension at a load of 15 kN \((P)\) : 0.045 mm
Ultimate load \((P_u)\) : 208.60 kN
Load at failure \((P_f)\) : 220 kN
Length of specimen after failure \((L_f)\) : 375 mm
Diameter at the end of failure \((d_f)\) : 17.75 mm

**Determine:**

- Young's modulus \((E)\)
- Proportionality limit \((\sigma_{pl})\)
- Ultimate stress \((\sigma_u)\)
- Percentage elongation
- Tru breaking stress \((\sigma_{ft})\)
- Percentage reduction in area
- Allowable stress when factor of safety is 2 \((\sigma_{allow})\)
- Modulus of resilience \((U_r)\)