THREE DIMENSIONAL SYSTEMS

‘3D - CARTESIAN VECTORS’
3-D Cartesian Vectors

- *Cartesian* \((x, y, z)\) Coordinate System
- Right-handed coordinate system
- Positive \(z\) axis points upwards, this axis helps us in measuring the height of an object or finding the altitude of any point.
Cartesian \((x, y, z)\) Coordinate System

Right-handed coordinate system

Various Orientations
3-D Cartesian Vectors

[Right-Handed Coordinate System]

A **rectangular** coordinate system is said to be **right-handed** provided:

Thumb of **right hand** points in the direction of the **positive z axis**. When the right-hand fingers are curled about this axis, the fingers direct from the **positive x axis** towards the **positive y axis**.
ESTABLISHING VECTOR IN 3D

- READING THE COORDINATES WITH RESPECT TO ANY REFERENCE
  (THE ORIGIN IN GENERAL)

- \( A = < x_A, y_A, z_A > \)

- \( B = < x_B, y_B, z_B > \)
Example:
Obtain the coordinates of the points A, B and C.
3-D Cartesian Vectors

Example:
Obtain the coordinates of the points A, B and C. Always give in the form of \textbf{point name} (x, y, z).

A (4, 2, -6)
B (0, 2, 0)
C (6, -1, 4)
3-D Cartesian Vectors

**Rectangular Components of a Vector**
A vector $\mathbf{A}$ may have **one, two or three** rectangular components along $x$, $y$ and $z$ axes, depending on its orientation.

By two successive application of the parallelogram law

$$\mathbf{A} = \mathbf{A}' + \mathbf{A}_z$$
$$\mathbf{A}' = \mathbf{A}_x + \mathbf{A}_y$$

Combining the equations, $\mathbf{A}$ can be expressed as:

$$\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y + \mathbf{A}_z$$
3-D Cartesian Vectors

• Unit Vector
  - Direction of $\mathbf{A}$ can be specified using a unit vector.
  - Unit vector has a magnitude of 1.
  - If $\mathbf{A}$ is a vector having a magnitude of $A \neq 0$, unit vector having the same direction as $\mathbf{A}$ is expressed by $\lambda_A = \mathbf{A} / A$

So that: $\mathbf{A} = A \lambda_A$
3-D Cartesian Vectors

- **Unit Vector**

  Magnitude $A$ has the same sets of units, hence the unit vector $\lambda_A$ is dimensionless. $A$ (a positive scalar) defines magnitude of $A$ where $\lambda_A$ defines: the direction and sense.
3-D Cartesian Vectors

Cartesian Unit Vectors

- Cartesian unit vectors, \( \mathbf{i} \), \( \mathbf{j} \), and \( \mathbf{k} \) are used to designate the directions of \( x \), \( y \), and \( z \) axes.

- Sense (or arrowhead) of these vectors are described by:
  - plus (+) or minus (-) sign,
  - depending on pointing towards the positive or negative axes.
3-D Cartesian Vectors

- Cartesian Vector Representations

Three components of \( \mathbf{A} \) act in the positive \( \mathbf{i} \), \( \mathbf{j} \) and \( \mathbf{k} \) directions

\[
\mathbf{A} = A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}
\]

*Note that the magnitude and direction of each component is separated, for simplifying the vectors algebraic operations.*
• Right-handed coordinate system:

Express vector $\lambda$ in terms of vector components $\lambda_x$, $\lambda_y$ & $\lambda_z$ parallel to the $x$, $y$ & $z$ axes respectively:

$$\lambda = \lambda_x + \lambda_y + \lambda_z$$
Position Vectors

Position vector $\mathbf{r}_{\text{OP}}$ is defined as: a fixed vector which locates a point in space relative to another point. In this case the origin $O$. 
Position Vectors

$r_{OP}$ extends from the origin, 'O $(x_o, y_o, z_o)$' which is $O (0, 0, 0)$ to point 'P $(x_p, y_p, z_p)$' then, in Cartesian vector form

$r_{OP} = (x_p - x_0)i + (y_p - y_0)j + (z_p - z_0)k$

$r_{OP} = x_p i + y_p j + z_p k$
Position Vectors

Note the tail to head vector addition of the three components

Start at origin O, one travels

x in the $+i$ direction,
y in the $+j$ direction and
z in the $+k$ direction,
arriving at point P $(x, y, z)$
ESTABLISHING POSITION VECTOR IN 3D between two interested points

READING THE COORDINATES WITH RESPECT TO ANY REFERENCE (THE ORIGIN IN GENERAL)

\[ A = \langle x_A, y_A, z_A \rangle \]

\[ B = \langle x_B, y_B, z_B \rangle \]
Finding POSITION VECTORE ‘r??’:

- Establish an arrow between the two interested points
- Give letter names to the tail and the head of this arrow
- The name of the line is: $r??$
- Obtain the coordinates of these two ends by getting the difference of the respective coordinates from head to tail.
Position Vectors

\( \mathbf{i}, \mathbf{j}, \mathbf{k} \) components of the positive vector \( \mathbf{r} \) may be formed by taking the coordinates of the tail, \( A (x_A, y_A, z_A) \) and subtracting them from the head \( B (x_B, y_B, z_B) \). i.e other than origin.

Note that:
the tail to head vector addition of the three components
The length ‘$r_{AB}$’

The distance between any desired two points can be obtained by writing the position vector between two points and calculating the magnitude of the vector. (as shown in the next example)

Recall that magnitude of a vector is:

$$r = \sqrt{(r_x)^2 + (r_y)^2 + (r_z)^2}$$
Example:
An elastic rubber band is attached to points A and B. Determine the distance between A and B.
The length \( r_{AB} \)

Coordinates

A \((1, 0, -3)\)

B \((-2, 1.5, 3)\)

1.5 m
Solution

Position vector
\[ r = [-2m - 1m]i + [1.5m - 0]j + [3m - (-3m)]k \]
\[ = \{-3.000 i + 1.500 j + 6.000 k\} m \]

Magnitude = length of the rubber band
\[ r = \sqrt{(-3.000)^2 + (1.500)^2 + (6.000)^2} = 6.874 \text{ m} \]

Unit vector in the director of \( r \)
\[ \lambda = \frac{r}{r} \]
\[ = \frac{-3.000}{6.874} i + \frac{1.500}{6.874} j + \frac{6.000}{6.874} k \]
WAYS OF FINDING THE COMPONENTS OF FORCES IN 3D

1. By **COORDINATES**
   (with the help of \( \lambda \))

2. By **ANGLES** given
   ➢ from any **AXIS**
     
     (cosine of this angle gives the component of that axis \( \alpha (\theta_x), \beta (\theta_y) \text{ and } \gamma (\theta_z) \))

   ➢ from **PLANE**

     \((\theta_{xy}, \theta_{xz} \text{ or } \theta_{yz}) \text{ use sine of the angles aswell})}
WAYS OF FINDING THE COMPONENTS OF FORCES IN 3D

1. By COORDINATES
   (with the help of $\lambda$)
The Lambda ‘λ’ Vector

- Force $\mathbf{F}$ acting along the chain can be presented as a Cartesian vector by establishing $x, y, z$ axes and forming a position vector $\mathbf{r}$ along length of chain.
Force Vector Directed along a Line

The Lambda ‘\( \lambda \)’ Vector

Unit vector, \( \lambda = \frac{r}{r} \) that defines the direction of both the chain and the force

- We get \( \mathbf{F} = F \lambda \)
Finding the Lambda vector ‘λ’:

Similarly, obtaining the vectorial components of the force ‘\( F \)’, the unit vector lambda ‘\( \lambda \)’ should be defined.

[In some books lambda is denoted by ‘\( u \)’].

\[
\vec{F} = F \vec{\lambda}
\]

\[
F(\text{vector}) = F(\text{scalar}) \cdot \lambda(\text{vector})
\]
\[ \vec{\lambda}_{AB} = \frac{\langle r_{AB} \rangle}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}} \]

\[ \vec{\lambda}_{AB} = \frac{\langle x_B - x_A \hat{i}, y_B - y_A \hat{j}, z_B - z_A \hat{k} \rangle}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}} \]

\[ \therefore \vec{F} = F \cdot \vec{\lambda}_{AB} = F \cdot \frac{\langle x_B - x_A \hat{i}, y_B - y_A \hat{j}, z_B - z_A \hat{k} \rangle}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}} \]
Force Vector Directed along a Line
The Lambda ‘λ’ Vector

Example:
The man pulls on the cord with a force of 350 N. Represent this force as a Cartesian vector components.
Force Vector Directed along a Line
The Lambda ‘$\lambda$’ Vector

Example:
The man pulls on the cord with a force of 350 N.
Represent this force as a Cartesian vector components.
Solution

End points of the cord are A (0m, 0m, 7.5m) and B (3m, -2m, 1.5m)

\[ \mathbf{r}_{AB} = (3m - 0m)i + (-2m - 0m)j + (1.5m - 7.5m)k \]

= \{3.000 i - 2.000 j - 6.000 k\} m

Magnitude = length of cord BA

\[ r = \sqrt{(3.000m)^2 + (-2.000m)^2 + (6.000m)^2} = 7.000m \]

Unit vector, \( \lambda = \frac{\mathbf{r}}{r} \)

= 3.000/7.000i - 2.000/7.000j - 6.000/7.000k

\[ \mathbf{F}_{AB} = 350 \left[ 3/7 \hat{i} - 2/7 \hat{j} - 6/7 \hat{k} \right] \]
Exercise:

Obtain the components of the cable that carries tension $T_{AB}$ of magnitude 62 kN.
Exercise:

A : <0, 18, 30>  B : <6, 13, 0>  $T_{AB} = 62 \text{kN}$

$$
\vec{\lambda}_{AB} = \frac{\langle (6 - 0) \hat{i}, (13 - 18 \hat{j}), (0 - 30 \hat{k}) \rangle}{\sqrt{(6)^2 + (-5)^2 + (-30)^2}}
$$

$$
\vec{\lambda}_{AB} = \frac{\langle 6 \hat{i}, -5 \hat{j}, -30 \hat{k} \rangle}{31}
$$

$$
T_{AB} = 62 \left[ \frac{6}{31} \hat{i} - \frac{5}{31} \hat{j} - \frac{30}{31} \hat{k} \right]
$$
WAYS OF FINDING THE COMPONENTS OF FORCES IN 3D

2. By **ANGLES** given

- from any **AXIS**

  (cosine of this angle gives the component of that axis $\alpha (\theta_x), \beta (\theta_y)$ and $\gamma (\theta_z)$)
Direction Cosines of Cartesian Vectors

- Force, \( \mathbf{F} \) that the tie down rope exerts on the ground support at \( O \) is directed along the rope.
- Angles \( \alpha, \beta \) and \( \gamma \) can be solved with axes \( x, y \) and \( z \).
• Cosines of their values forms a unit vector \( \lambda \) that acts in the direction of the rope

• Force \( \mathbf{F} \) has a magnitude of \( F \)

\[
\mathbf{F} = F \, \lambda = F \cos \alpha \mathbf{i} + F \cos \beta \mathbf{j} + F \cos \gamma \mathbf{k}
\]
3-D Cartesian Vectors Directions

- **Direction of a Cartesian Vector**
  - All angles measured from **COORDINATE AXES**
  - Orientation of A is defined as the coordinate direction angles $\alpha (\theta_x)$, $\beta (\theta_y)$ and $\gamma (\theta_z)$ measured between the tail of A and the positive x, y and z axes.
    
    $0^\circ \leq \alpha \leq 180^\circ$
    $0^\circ \leq \beta \leq 180^\circ$
    $0^\circ \leq \gamma \leq 180^\circ$
3-D Cartesian Vectors Directions

• **Direction of a Cartesian Vector**
  
  For angles $\alpha (\theta_x)$, (blue colored triangle), one can calculate the *direction cosines* of $\mathbf{A}$

\[
\cos \alpha = \frac{A_x}{A}
\]
3-D Cartesian Vectors Directions

• Direction of a Cartesian Vector

For angle $\beta (\theta_y)$ (blue colored triangle), one can calculate the *direction cosines* of $A$

$$\cos \beta = \frac{A_y}{A}$$
3-D Cartesian Vectors Directions

• Direction of a Cartesian Vector

For angle $\gamma (\theta_z)$ (blue colored triangles), one can calculate the direction cosines of $\mathbf{A}$

$$\cos \gamma = \frac{A_z}{A}$$
3-D Cartesian Vectors Directions

• Direction of a Cartesian Vector

Vector $\mathbf{A}$ expressed in Cartesian vector form:

$$\mathbf{A} = A \lambda_A$$

$$= A \cos \alpha \mathbf{i} + A \cos \beta \mathbf{j} + A \cos \gamma \mathbf{k}$$

$$= A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}$$
3-D Cartesian Vectors Directions

Direction of a Cartesian Vector

Angles $\alpha$, $\beta$ and $\gamma$ can be determined by the inverse cosines:

\[
\cos \alpha = \frac{A_x}{A}
\]
\[
\cos \beta = \frac{A_y}{A}
\]
\[
\cos \gamma = \frac{A_z}{A}
\]
3-D Cartesian Vectors Directions

• **Direction of a Cartesian Vector**

\[ \lambda_A = \cos \alpha \, \mathbf{i} + \cos \beta \, \mathbf{j} + \cos \gamma \, \mathbf{k} \]

Since \[ A = \sqrt{A_x^2 + A_y^2 + A_z^2} \]

magnitude of \( \lambda_A = 1 \),

\[ \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1^2 = 1 \]
Direction Cosines of Cartesian Vectors

\[
\cos \theta_x = \cos \alpha = \frac{\langle x_B - x_A \hat{i} \rangle}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}
\]

\[
\cos \theta_y = \cos \beta = \frac{\langle y_B - y_A \hat{j} \rangle}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}
\]

\[
\cos \theta_z = \cos \gamma = \frac{\langle z_B - z_A \hat{k} \rangle}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}
\]
Direction Cosines of Cartesian Vectors

Example:

Determine the components of the force \( F = 200 \).
Direction Cosines of Cartesian Vectors

Since only 2 angles are given, the 3\textsuperscript{rd} angle will be determined by:

\[
\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1
\]
\[
\cos^2 \alpha + \cos^2 43^\circ + \cos^2 55^\circ = 1
\]
\[
\cos \alpha = \sqrt{1 - (0.7314)^2 - (0.5736)^2} = \pm 0.3690
\]

where there are two possibilities; either

\[
\alpha = \cos^{-1}(0.3690) = 68.3485^\circ
\]
or

\[
\alpha = \cos^{-1}(-0.3690) = 111.6515^\circ
\]
By inspection we observed that $\alpha = 68.3485^\circ$ since component of $F_x$ is at $+x$ axis direction.

Hence $F = 200$ N

$$F = F \cos \alpha \mathbf{i} + F \cos \beta \mathbf{j} + F \cos \gamma \mathbf{k}$$

$$= (200 \cos 68.3548^\circ \text{N}) \mathbf{i} + (200 \cos 43^\circ \text{N}) \mathbf{j} + (200 \cos 55^\circ \text{N}) \mathbf{k}$$

$$= \{73.760 \mathbf{i} + 146.271 \mathbf{j} + 114.715 \mathbf{k}\} \text{ N}$$

**Control:**

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2}$$

$$= \sqrt{(73.760)^2 + (146.271)^2 + (114.715)^2} = 200 \text{ N}$$
The man pulls on the cord with a force of 350 N. Determine the direction cosines of this force.
Exercise:
The man pulls on the cord with a force of 350 N. Determine the direction cosines of this force.
Direction Cosines of Cartesian Vectors

Solution

End points of the cord are A (0m, 0m, 7.5m) and B (3m, -2m, 1.5m)

\[ \mathbf{r}_{AB} = (3m - 0m)i + (-2m - 0m)j + (1.5m - 7.5m)k \]

\[ = \{3.000 \mathbf{i} - 2.000 \mathbf{j} - 6.000 \mathbf{k}\} \text{ m} \]

Magnitude = length of cord BA

\[ r = \sqrt{(3.000m)^2 + (2.000m)^2 + (6.000m)^2} = 7.000m \]

Unit vector, \( \lambda = \mathbf{r} / r \)

\[ = 3.000/7.000 \mathbf{i} - 2.000/7.000 \mathbf{j} - 6.000/7.000 \mathbf{k} \]
The magnitude of the force $F$ is $350$ N, so with $\lambda$ unit vector components

$$F = F\lambda$$

$$= 350\text{ N} \left(\frac{3.000}{7.000} \mathbf{i} - \frac{2.000}{7.000} \mathbf{j} - \frac{6.000}{7.000} \mathbf{k}\right)$$

$$= \{150.000 \mathbf{i} - 100.000 \mathbf{j} - 300.000 \mathbf{k}\} \text{ N}$$

Angles are:

$$\alpha = \cos^{-1}\left(\frac{3}{7}\right) = 64.62^\circ$$

$$\beta = \cos^{-1}\left(-\frac{2}{7}\right) = 106.60^\circ$$

$$\gamma = \cos^{-1}\left(-\frac{6}{7}\right) = 149^\circ$$
Exercise:
The coordinates of point $C$ of the truss are:
$x_C = 4 \text{ m}, \ y_C = 0, \ z_C = 0,$ and
the coordinates of point $D$ are:
$x_D = 2 \text{ m}, \ y_D = 3 \text{ m}, \ z_D = 1 \text{ m}.$

What are the Coordinate direction angles
of the position vector $\mathbf{r}_{CD}$ (from point C to point D)?
Coordinate direction angles of Cartesian Vectors

Knowing the coordinates of points $C$ and $D$, one can determine $\mathbf{r}_{CD}$ in terms of its components.

Then the magnitude of $\mathbf{r}_{CD}$ (the distance from $C$ to $D$) can be calculated using the direction cosines.
The position vector $\mathbf{r}_{CD}$ in terms of its components.

$$\mathbf{r}_{CD} = (x_D - x_C)\mathbf{i} + (y_D - y_C)\mathbf{j} + (z_D - z_C)\mathbf{k}$$

$$= (2 - 4)\mathbf{i} + (3 - 0)\mathbf{j} + (1 - 0) \mathbf{k} \text{ (m)}$$

$$= -2\mathbf{i} + 3\mathbf{j} + \mathbf{k} \text{ (m)}$$

$$|\mathbf{r}_{CD}| = \sqrt{\mathbf{r}_{CDx}^2 + \mathbf{r}_{CDy}^2 + \mathbf{r}_{CDz}^2}$$

$$= \sqrt{(-2 \text{ m})^2 + (3 \text{ m})^2 + (1 \text{ m})^2}$$

$$= 3.742 \text{ m}$$
Coordinate direction angles of Cartesian Vectors

\[
\cos \alpha = \frac{\mathbf{r}_{CDx}}{|\mathbf{r}_{CD}|} = \frac{-2 \text{ m}}{3.742 \text{ m}} = 0.5345
\]
\[
\alpha = 122.3100^\circ
\]

\[
\cos \beta = \frac{\mathbf{r}_{CDy}}{|\mathbf{r}_{CD}|} = \frac{3 \text{ m}}{3.742 \text{ m}} = 0.8017
\]
\[
\beta = 36.7073^\circ
\]

\[
\cos \gamma = \frac{\mathbf{r}_{CDz}}{|\mathbf{r}_{CD}|} = \frac{1 \text{ m}}{3.742 \text{ m}} = 0.2672
\]
\[
\gamma = 74.5023^\circ
\]
Solution

Determine the coordinate direction angles of Cartesian Vector $\vec{r}_{CD}$.

- $\alpha = 122.3100$°
- $\beta = 36.7073$°
- $\gamma = 74.5023$°
Exercise:
For the given elastic rubber band AB, determine its direction.
Direction Cosines of Cartesian Vectors

Solution

Position vector

\[ r = [-2m - 1m]i + [2m - 0]j + [3m - (-3m)]k \]
\[ = \{-3.000 i + 2.000 j + 6.000 k\} \text{ m} \]

Magnitude = length of the rubber band

\[ r = \sqrt{(-3.000)^2 + (2.000)^2 + (6.000)^2} = 7.000 \text{ m} \]

Unit vector in the director of \( r \)

\[ \lambda = \frac{r}{r} \]
\[ = -3.000/7.000i + 2.000/7.000j + 6.000/7.000k \]
Direction Cosines of Cartesian Vectors

Coordinates

A (1, 0, -3)

B (-2, 2, 3)
Direction of Cartesian Vectors

\[ \alpha = \cos^{-1}\left(-\frac{3.000}{7.000}\right) = 115.38^\circ \]
\[ \beta = \cos^{-1}\left(\frac{2.000}{7.000}\right) = 73.40^\circ \]
\[ \gamma = \cos^{-1}\left(\frac{6.000}{7.000}\right) = 31^\circ \]
WAYS OF FINDING THE COMPONENTS OF FORCES IN 3D

2. By **ANGLES** given

- **from PLANE**
  
  \[ \left( \theta_{xy}, \theta_{xz}, \text{ or } \theta_{yz} \right) \text{ use sine of the angles as well} \]
3-D Cartesian Vectors Directions

- **Direction of a Cartesian Vector**

  ii. Angle measured from coordinate **PLANE**

\[
A_z = A \cos \gamma
\]

\[
A' = A \sin \gamma
\]

\[
A_x = A' \cos \theta_{xy} = A \sin \gamma \cos \theta_{xy}
\]

\[
A_y = A' \sin \theta_{xy} = A \sin \gamma \sin \theta_{xy}
\]
Example:
Determine the components of the vector \( \mathbf{A} = 5 \) units.
Example:
Determine the components of the vector $\mathbf{A}=5$ units.

Be careful about $x$-$y$-$z$ coordinates!

$$A_y = 5 \cos 22 = 4.636 \text{ units}$$
$$A_z = 5 \sin 22 \cos 37 = 1.496 \text{ units}$$
$$A_x = 5 \sin 22 \sin 37 = 1.127 \text{ units}$$
Addition and Subtraction of Cartesian Vectors

Example
Given: \( \mathbf{A} = A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k} \)
and \( \mathbf{B} = B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k} \)

**Vector Addition**
Resultant \( \mathbf{R} = \mathbf{A} + \mathbf{B} \)
\[
= (A_x + B_x) \mathbf{i} + (A_y + B_y) \mathbf{j} + (A_z + B_z) \mathbf{k}
\]

**Vector Subtraction**
Resultant \( \mathbf{R} = \mathbf{A} - \mathbf{B} \)
\[
= (A_x - B_x) \mathbf{i} + (A_y - B_y) \mathbf{j} + (A_z - B_z) \mathbf{k}
\]
Addition and Subtraction of Cartesian Vectors

**Concurrent Force resultant** is the vector sum of all the forces in the system

\[ \mathbf{F}_R = \sum \mathbf{F} = \sum F_x \mathbf{i} + \sum F_y \mathbf{j} + \sum F_z \mathbf{k} \]

where:

\[ \sum F_x, \sum F_y \text{ and } \sum F_z \]

represent the algebraic sums of the \( x, y \) and \( z \) or \( i, j \) and \( k \) components of each force in the system.
Addition and Subtraction of Cartesian Vectors

Example:
Determine the magnitude and coordinate direction angles of resultant force acting on the ring.

\[ F_2 = (50\hat{i} - 100\hat{j} + 100\hat{k}) \text{kN} \quad F_1 = (60\hat{j} + 80\hat{k}) \text{kN} \]
Addition and Subtraction of Cartesian Vectors

Solution

Resultant force

\[ \mathbf{F}_R = \sum \mathbf{F} \]

\[ = \mathbf{F}_1 + \mathbf{F}_2 \]

\[ = \{60 \mathbf{j} + 80 \mathbf{k}\} \text{ kN} \]

\[ + \{50 \mathbf{i} - 100 \mathbf{j} + 100 \mathbf{k}\} \text{ kN} \]

\[ = \{50.000 \mathbf{i} - 40.000 \mathbf{j} + 180.000 \mathbf{k}\} \text{ kN} \]

Magnitude of \( \mathbf{F}_R \) is found by

\[ F_R = \sqrt{(50.000)^2 + (40.000)^2 + (180.000)^2} \]

\[ = 191.050 \text{ kN} \]
Addition and Subtraction of Cartesian Vectors

Solution

Unit vector acting in the direction of $\mathbf{F}_R$

$$\lambda_{FR} = \frac{\mathbf{F}_R}{F_R} = \frac{50.000}{191.050} \mathbf{i} + \frac{-40.000}{191.050} \mathbf{j} + \frac{180.000}{191.050} \mathbf{k} = 0.2617 \mathbf{i} - 0.2094 \mathbf{j} + 0.9422 \mathbf{k}$$

So that

$$\cos \alpha = 0.2617 \quad \alpha = 74.8290^\circ$$
$$\cos \beta = -0.2094 \quad \beta = 102.0872^\circ$$
$$\cos \gamma = 0.9422 \quad \gamma = 19.5756^\circ$$
Addition and Subtraction of Cartesian Vectors

\[ \mathbf{F}_R = (50\mathbf{i} - 40\mathbf{j} + 180\mathbf{k}) \text{kN} \]

\[ \alpha = 74.8290^\circ \]

\[ \beta = 102.0872^\circ \]

\[ \gamma = 19.5756^\circ \]
### Addition and Subtraction of Cartesian Vectors

**Example:** The roof is supported by cables. If the cables exert $\mathbf{F}_{AB} = 100 \, \text{N}$ and $\mathbf{F}_{AC} = 120 \, \text{N}$ on the wall hook at A, determine the magnitude of the resultant force acting at A.
Addition and Subtraction of Cartesian Vectors

Solution

\[ \mathbf{r}_{AB} = (4 \text{ m} - 0 \text{ m}) \mathbf{i} + (0 \text{ m} - 0 \text{ m}) \mathbf{j} + (0 \text{ m} - 4 \text{ m}) \mathbf{k} \]

\[ = \{4.000 \mathbf{i} - 4.000 \mathbf{k}\} \text{ m} \]

\[ \mathbf{r}_{AB} = \sqrt{(4.000 \text{ m})^2 + (-4.000 \text{ m})^2} = 5.657 \text{ m} \]

\[ \mathbf{F}_{AB} = 100 \text{ N} \left( \mathbf{r}_{AB} / r_{AB} \right) \]

\[ = 100 \text{ N} \left\{ \left(\frac{4}{5.657}\right) \mathbf{i} - \left(\frac{4}{5.657}\right) \mathbf{k} \right\} \]

\[ = \{70.711 \mathbf{i} - 70.711 \mathbf{k}\} \text{ N} \]
Addition and Subtraction of Cartesian Vectors

Solution

\[ \mathbf{r}_{AC} = (4m - 0m) \mathbf{i} + (2m - 0m) \mathbf{j} + (0m - 4m) \mathbf{k} \]

\[ = \{4.000 \mathbf{i} + 2.000 \mathbf{j} - 4.000 \mathbf{k}\} \text{ m} \]

\[ r_{AC} = \sqrt{(4.000 \text{ m})^2 + (2.000 \text{ m})^2 + (4.000 \text{ m})^2} = 6.000 \text{ m} \]

\[ \mathbf{F}_{AC} = 120\text{N} \left( \mathbf{r}_{AC}/r_{AC} \right) \]

\[ = 120\text{N} \{ (4.000/6.000) \mathbf{i} + (2.000/6.000) \mathbf{j} - (4.000/6.000) \mathbf{k}\} \]

\[ = \{80.000 \mathbf{i} + 40.000 \mathbf{j} - 80.000 \mathbf{k}\} \text{ N} \]
Addition and Subtraction of Cartesian Vectors

Solution

\[ \mathbf{F}_R = \mathbf{F}_{AB} + \mathbf{F}_{AC} \]
\[ = \{70.711 \mathbf{i} - 70.711 \mathbf{k}\} \text{ N} + \{80.000 \mathbf{i} + 40.000 \mathbf{j} - 80.000 \mathbf{k}\} \text{ N} \]
\[ = \{150.711 \mathbf{i} + 40.000 \mathbf{j} - 150.711 \mathbf{k}\} \text{ N} \]

Magnitude of \( \mathbf{F}_R \):

\[ \mathbf{F}_R = \sqrt{(150.711)^2 + (40.000)^2 + (-150.711)^2} \]
\[ = 216.859 \text{ N} \]
Direction Cosines of Cartesian Vectors

Example:

\[ \begin{align*}
&\vec{F}_1 = 2 \text{ kN} \\
&\vec{F}_2 = 5 \text{ kN} \\
&\text{angle } 45^\circ \quad \text{angle } 49.2^\circ \quad \text{angle } 74.3^\circ
\end{align*} \]

a) Express each force as a Cartesian vector.
b) Determine the resultant force \( \vec{F}_R \).
c) Find the magnitude and coordinate direction angles of the resultant force.
Example:
Find the magnitude and direction of the resultant of the two forces shown if \( P = 300 \) N and \( Q = 400 \) N.
Exercise:
Determine the magnitude and coordinate direction angles of resultant force.

Addition and Subtraction of Cartesian Vectors
Example:

a) Determine the resultant force $\mathbf{R}$ of the given 3 forces (OA, OB and OC) knowing that force OA has positive component along X axis.
b) Find the coordinate direction angles ($\theta_x$, $\theta_y$, $\theta_z$) of this resultant $\mathbf{R}$. 

![Diagram showing forces OA, OB, and OC with their respective angles and magnitudes.](image-url)
Chapter Summary

Parallelogram Law

• Addition of two vectors
• Components form the side and resultant form the diagonal of the parallelogram
• To obtain resultant, use tip to tail addition by triangle rule
• To obtain magnitudes and directions, use Law of Cosines and Law of Sines
Chapter Summary

Cartesian Vectors

- Vector $\mathbf{F}$ resolved into Cartesian vector form
  \[ \mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k} \]
- Magnitude of $\mathbf{F}$
  \[ F = \sqrt{F_x^2 + F_y^2 + F_z^2} \]
- Coordinate direction angles $\alpha$, $\beta$ and $\gamma$ are determined by the formulation of the unit vector in the direction of $\mathbf{F}$
  \[ \mathbf{u} = \left( \frac{F_x}{F} \right) \mathbf{i} + \left( \frac{F_y}{F} \right) \mathbf{j} + \left( \frac{F_z}{F} \right) \mathbf{k} \]
Chapter Summary

**Cartesian Vectors**
- Components of \( \lambda \) represent \( \cos \alpha, \cos \beta \) and \( \cos \gamma \)
- These angles are related by
  \[
  \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1
  \]

**Force and Position Vectors**
- Position Vector is directed between 2 points
- Formulated by distance and direction moved along the \( x, y \) and \( z \) axes from tail to tip
Chapter Summary

**Force and Position Vectors**

- For line of action through the two points, it acts in the same direction of \( \lambda \) as the position vector.
- Force expressed as a Cartesian vector
  \[
  \mathbf{F} = F \mathbf{\lambda} = F \left( \mathbf{r}/r \right)
  \]
| **A scalar** is a positive or negative number; e.g., mass and temperature. |
| A vector has a magnitude and an arrowhead sense of direction; e.g., force and position. *(See page 17.)* |
| Multiplication or division of a vector by a scalar will change only the magnitude of the vector. If the scalar is negative the sense of the vector will change, so that it acts in the opposite direction. *(See page 18.)* |
| If vectors are collinear, the resultant is formed by an algebraic or scalar addition. *(See page 19.)* |

\[ R = A + B \]
Parallelogram Law

Two forces add according to the parallelogram law. The *components* form the sides of the parallelogram and the *resultant* is the diagonal.

To find the components of a force along any two axes, extend lines from the head of the force, parallel to the axes, to form the components.

To obtain the components or the resultant, show how the forces add by a tip-to-tail addition using the triangle rule, and then use the law of sines and the law of cosines to calculate their values. *(See pages 18–21.)*
Rectangular Components: Two Dimensions

Vectors $F_x$ and $F_y$ are rectangular components of $F$.

The resultant force is determined from the algebraic sum of its components. (See page 31.)

\[
F_{Rx} = \sum F_x \\
F_{Ry} = \sum F_y \\
F_R = \sqrt{(F_{Rx})^2 + (F_{Ry})^2}, \quad \theta = \tan^{-1}\left|\frac{F_{Ry}}{F_{Rx}}\right|
\]
Chapter Review

Cartesian Vectors
The unit vector $\mathbf{u}_A$ has a length 1, no units, and it points in the direction of the vector $\mathbf{F}$.

A force can be resolved into its Cartesian components along the $x, y, z$ axes so that $\mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k}$.

The magnitude of $\mathbf{F}$ is determined from the positive square root of the sum of the squares of its components.

The coordinate direction angles $\alpha, \beta, \gamma$ are determined by formulating a unit vector in the direction of $\mathbf{F}$. The components of $\mathbf{u}$ represent $\cos \alpha, \cos \beta, \cos \gamma$.

$$u = \frac{\mathbf{F}}{F}$$

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2}$$

$$u = \frac{F_x}{F} \mathbf{i} + \frac{F_y}{F} \mathbf{j} + \frac{F_z}{F} \mathbf{k}$$

$$u = \cos \alpha \mathbf{i} + \cos \beta \mathbf{j} + \cos \gamma \mathbf{k}$$
The coordinate direction angles are related so that only two of the three angles are independent of one another.

\[ \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \]

To find the resultant of a concurrent force system, express each force as a Cartesian vector and add the i, j, k components of all the forces in the system.

*(See pages 43–46.)*

**Position and Force Vectors**

A position vector locates one point in space relative to another. The easiest way to formulate the components of a position vector is to determine the distance and direction that one must travel along the x, y, and z directions—going from the tail to the head of the vector.

\[ r = (x_B - x_A)\hat{i} + (y_B - y_A)\hat{j} + (z_B - z_A)\hat{k} \]

If the line of action of a force passes through points A and B, then the force acts in the same direction \( \mathbf{u} \) as the position vector \( \mathbf{r} \). The force can then be expressed as a Cartesian vector.

*(See pages 55, 56, and 58.)*
Finding the length ‘$r_{AB}$’

(a)

(b)
OBTAINING THE DIRECTION VECTOR COMPONENTS

\[ \mathbf{r} = (x_{\text{Head-Tail}}, y_{\text{Head-Tail}}, z_{\text{Head-Tail}}) \]

While reading use: tail \quad head

\[ \text{arrow head coordinate} - \text{arrow tail coordinate} \]
Finding the length ‘r’:

• Read the coordinates of the two ends of the line
• Get the difference of the coordinates from the tail of the force coordinate to the point of interest.

OBTAIN THE DIRECTION VECTOR COMPONENTS

\[ \mathbf{r} (x_{\text{Head-Tail}}, y_{\text{Head-Tail}}, z_{\text{Head-Tail}}) \]

While reading use: tail → head

<arrow head coordinate – arrow tail coordinate>